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Complex Oscillation and Removable Sets

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Let A be a transcendental entire function of finite order, and let E be the product of linearly independent solutions of $w'' + A(z)w = 0$. We prove the existence of sequences of annuli Ω_m such that if E has relatively few zeros in the union of the Ω_m, then E has relatively few zeros in the whole plane. © 1999 Academic Press

1. INTRODUCTION

Let A be an entire function, and let f_1, f_2 be linearly independent solutions of the equation

$$w'' + A(z)w = 0, \tag{1}$$

normalized so that the Wronskian $W = W(f_1, f_2) = f_1 f_2' - f_1' f_2$ satisfies $W = 1$. The Bank–Laine product function $E = f_1 f_2$ satisfies $E'(z) = \pm 1$ at every zero z of E, as well as the relation

$$4A = (E'/E)^2 - 2E''/E - 1/E^2. \tag{2}$$

Conversely, if E is any entire function with the property that $E'(z) = \pm 1$ at every zero z of E, then [2] the function A defined by (2) is entire, and E is the product of linearly independent normalized solutions of (1).

Extensive work in recent years has concerned the exponent of convergence $\lambda(f_j)$ of the zeros of solutions f_j, in connection with the order of

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growth $\rho(A)$ of the coefficient A, these defined by

$$\lambda(f_j) = \limsup_{r\to\infty} \frac{\log^+ N(r, 1/f_j)}{\log r}, \qquad \rho(A) = \limsup_{r\to\infty} \frac{\log^+ T(r, A)}{\log r}.$$

Note that

$$\rho(E) \geq \lambda(E) = \max\{\lambda(f_1), \lambda(f_2)\}.$$

It has been conjectured that the condition

$$A \text{ transcendental}, \quad \rho(A) < \infty, \qquad \lambda(E) < \infty \tag{3}$$

implies that $\rho(A)$ is a positive integer, and it has been shown [14, 15] that (3) implies that $\rho(A) > 1/2$ and that E has finite order [1]. Further results may be found in [3, 4, 10, 13] and elsewhere.

The present paper is concerned with a problem first considered in [11], that of the existence of sets D which are removable in the following sense: if A has finite order and the zeros of E outside D have a finite exponent of convergence then $\rho(E)$ is finite, and hence so is $\lambda(E)$.

THEOREM A [11]. *Let η, K, and S be constants with $0 < \eta < \pi$ and $1 < K < S$. Let R_m be a positive sequence tending to infinity with $R_{m+1} > SR_m$ for each positive integer m, and let ϕ_m be a real sequence. Let D be the union of the*

$$D_m = \{z = re^{i\theta}:\ R_m < r < KR_m,\ \phi_m - \eta < \theta < \phi_m + \eta\}.$$

Suppose that A is a transcendental entire function of finite order and that $E = f_1 f_2$ is the product of linearly independent normalized solutions f_j of (1), *and suppose finally that the zeros of E in the complement of D have finite exponent of convergence. Then E has finite order.*

Thus if E has relatively few zeros outside D then E has relatively few in the whole plane. A drawback of Theorem A is that the complement of the removable set D needs to be connected in order for the proof in [11], which is based in part on the methods of [18], to work. A further result [11, Theorem 3] eliminated this restriction, but subject to the strong additional hypothesis that there exists at least one ray $\arg z = \theta$ along which $A(z)$ has polynomial growth, i.e., $\log^+|A(re^{i\theta})| = O(\log r)$. We show here that the logarithmic rectangles occurring in Theorem A may be replaced by annuli, with no hypothesis on the coefficient function A other than that A is transcendental of finite order, the resulting removable set having disconnected complement. Our approach is more direct than that of [11], using in part a simplified version of the method of [12].

THEOREM 1. *Suppose that K and M are positive constants with $K > 1$, that A is a transcendental entire function of finite order, and that $E = f_1 f_2$ is the product of linearly independent normalized solutions of* (1). *Suppose that there exists a positive sequence r_m tending to infinity such that for each large positive integer m the number of zeros of E in the annulus*

$$\Omega(r_m, K) = \{z : K^{-1} r_m < |z| < K r_m\}$$

is at most $(r_m)^M$. Then

$$\log T\left(K^{-1} r_m, E\right) = O(\log r_m), \qquad m \to \infty. \tag{4}$$

THEOREM 2. *Suppose that K, M, A, E, r_m are as in the hypotheses of Theorem* 1, *and assume in addition that*

$$\limsup_{m \to \infty} \frac{\log r_{m+1}}{\log r_m} < \infty. \tag{5}$$

Then E has finite order.

Thus with the hypotheses of Theorem 2, the complement of the union of the $\Omega(r_m, K)$ is a removable set in the sense described above. The hypothesis in Theorems 1 and 2 that A has finite order is not redundant. Let H be an entire function, having $\exp(2^m)$ simple zeros on the circle $|z| = 2^m$, for each positive integer m, and no other zeros. As in [16] choose, using Mittag-Leffler interpolation, an entire function g such that $E = He^g$ satisfies $E'(z) = \pm 1$ at each of these zeros. Then E is the product of linearly independent normalized solutions of (1), and $\lambda(E) = \infty$, although E has no zeros in the annuli $2^m < |z| < 2^{m+1}$.

2. LEMMAS NEEDED FOR THE PROOF OF THEOREM 1

LEMMA 1. *Suppose that K, M, A, E, r_m are as in the hypotheses of Theorem* 1. *Then there exist positive constants M_1, M_2 with the following properties.*

If m is a sufficiently large positive integer there exists v_m in $\Omega(r_m, K^{1/4})$ such that

$$|\log|E(v_m)|| \leq (r_m)^{M_1}, \tag{6}$$

and such that E has no zeros in the disc $B(v_m, (r_m)^{-M_2})$.

Proof. We use c to denote a positive constant not depending on m, not necessarily the same at each occurrence. We note first that, by [12, p. 508], or using Herold's comparison theorem [7], the normalized solutions f_j of (1) satisfy, for large r,

$$|f_j(z)| + |f_j'(z)| \leq \exp\left(3rM(r, A)^{1/2}\right), \qquad |z| \leq r. \tag{7}$$

We may choose s with $K^{-1/8}r_m < s < K^{1/8}r_m$, such that s is normal for A with respect to the Wiman–Valiron theory [6, 17]. This means that if $|z_0| = s$ and $|A(z_0)| = M(s, A)$, then we have

$$A(z) = (z/z_0)^N A(z_0)(1 + o(1)) \tag{8}$$

and

$$A'(z)/A(z) = (1 + o(1))N/z, \qquad A''(z)/A(z) = (1 + o(1))N^2/z^2 \tag{9}$$

for z in $D(z_0, 2)$, in which

$$D(z_0, L) = \left\{z = z_0 e^{\tau} \colon |\mathrm{Re}(\tau)| \leq LN^{-5/8}, |\mathrm{Im}(\tau)| \leq LN^{-5/8}\right\}. \tag{10}$$

Here $N = \nu(s, A)$ is the central index of A and, provided s lies outside a set of finite logarithmic measure, may be assumed to satisfy

$$N \leq (\log M(s, A))^{5/4}. \tag{11}$$

Define

$$z_1 = z_0 \exp(-N^{-5/8}), \qquad Z = 2A(z_1)^{1/2} z_1/(N + 2) + \int_{z_1}^{z} A(t)^{1/2}\, dt. \tag{12}$$

We may write (8) in the form

$$A(z) = (z/z_1)^N A(z_1)(1 + \mu(z))^2, \qquad \mu(z) = o(1),$$

for z in $D(z_0, 2)$, so that $\mu'(z) = o(N^{5/8}s^{-1})$ for z in $D(z_0, 1)$. Thus for z in $D(z_0, 1)$, integration by parts from z_1 to z along part of the ray $\arg t = \arg z_1$ and part of the circle $|t| = |z|$, this path having length $O(sN^{-5/8})$, gives

$$\begin{aligned}
&\int_{z_1}^{z} t^{N/2}\mu(t)\, dt \\
&\qquad = o\left((N + 2)^{-1}|z|^{(N+2)/2}\right) - \int_{z_1}^{z} 2(N + 2)^{-1} t^{(N+2)/2} o(N^{5/8}s^{-1})\, dt \\
&\qquad = o\left((N + 2)^{-1}|z|^{(N+2)/2}\right).
\end{aligned}$$

Hence

$$Z = (1 + o(1))A(z_1)^{1/2} 2 z^{(N+2)/2}(z_1)^{-N/2}(N+2)^{-1}$$
$$= (1 + o(1))A(z)^{1/2} 2z(N+2)^{-1}, \tag{13}$$

for z in $D(z_0, 1)$. Further,

$$Z(z)/Z(z_0) = (1 + o(1))(z/z_0)^{(N+2)/2}$$

for z in $D(z_0, 1)$ and, since Z is locally univalent, by (12), the function Z has, in $D(z_0, 1/2)$, at least one simple island H_0 mapped univalently onto the closed region H_1 given by

$$\begin{gathered} |\log|Z/Z_0|| \leq N^{1/3}, \qquad |\arg Z| \leq \pi/4, \\ Z_0 = |A(z_0)^{1/2} 2 z_0 (N+2)^{-1}| = (1 + o(1))|Z(z_0)|. \end{gathered} \tag{14}$$

By (11), $Z_0 \exp(-N^{1/3})$ is large, when s is large enough.

As the next step in the proof of Lemma 1 we apply a local analogue of Hille's asymptotic method [8, 9] developed in [12]. We write

$$W(Z) = A(z)^{1/4} w(z), \tag{15}$$

for z in H_0 and Z in H_1, in which w is a solution of (1). The equation (1) transforms to

$$\begin{gathered} \frac{d^2W}{dZ^2} + (1 - F_0(Z))W = 0, \\ F_0(Z) = A''(z)/4A(z)^2 - 5A'(z)^2/16A(z)^3. \end{gathered} \tag{16}$$

By (9), we have $|F_0(Z)| \leq 3|Z|^{-2}$ in H_1. By [12, Lemma 1] there exist solutions $U_1(Z), U_2(Z)$ of (16) satisfying

$$\begin{gathered} U_j(Z) = (1 + o(1))\exp\big((-1)^j iZ\big), \\ U_j'(Z) = (1 + o(1))(-1)^j i \exp\big((-1)^j iZ\big), \\ W(U_1, U_2) = 2i + o(1) \end{gathered} \tag{17}$$

in H_1. We write, in H_0, for $q = 1, 2$,

$$f_q(z) = C_q u_1(z) + D_q u_2(z), \qquad u_j(z) = A(z)^{-1/4} U_j(Z), \tag{18}$$

with C_1, D_1, C_2, D_2 constants. Choose z^* in H_0 so that $Z^* = Z(z^*)$ satisfies

$$|Z^*| \le (1/2) Z_0, \qquad |U_2(Z^*)| \le 2, \qquad |U_2'(Z^*)| \le 2.$$

Then using (9), (11), (14), (17), and (18) we have

$$\begin{gathered} |z^*| \le s, \qquad u_2(z^*) = o(1), \\ |u_2'(z^*)/u_2(z^*)| \le N + 2M(s, A)^{1/2} \le 3M(s, A)^{1/2}. \end{gathered} \tag{19}$$

Further, (17) and standard properties of Wronskians give

$$W(u_1, u_2) = 2i + o(1)$$

in H_0. Thus the equation

$$C_1 = W(f_1, u_2)/W(u_1, u_2)$$

and (7) and (19) give

$$|C_1| \le cM(s, A)^{1/2} \exp\left(3sM(s, A)^{1/2}\right) \le M = \exp(4NZ_0), \tag{20}$$

using (14), and the same estimate holds for C_2, D_1, D_2.

We require further estimates for the coefficients C_1, C_2, D_1, D_2 and for convenience we state and prove these as Lemma 2, following which the proof of Lemma 1 will be completed.

LEMMA 2. *In each pair* $\{C_1, D_1\}, \{C_2, D_2\}$, *at least one term has modulus at most* M^{-2}.

Proof. Suppose that C_1 and D_1 each have modulus at least M^{-2}. Set $F_1(Z) = A(z)^{1/4} f_1(z)$, for z in H_0, and Z in H_1. Then we may write, using (18) and (20),

$$F_1(Z) = -C_1 U_1(Z)(e^{2iY} - 1), \qquad Y = Z + S + o(1), \qquad |S| \le 32 N Z_0. \tag{21}$$

It follows from (14), (20), and (21) that the image of H_1 under Y covers the region

$$(1/4) N^{1/3} \le \log|Y/Z_0| \le (1/2) N^{1/3}, \qquad |\arg Y| \le \pi/8,$$

so that the number of zeros of f_1 in H_0 is at least $\exp(cN^{1/3} Z_0)$. Since every zero of f_1 is a zero of E and since H_0 is contained in $\Omega(r_m, K)$, this contradicts the hypotheses of Theorem 1. Lemma 2 is proved.

We return to the proof of Lemma 1. Since F_1F_2 has at most $(r_m)^M$ zeros in H_1, we may choose Z_1 in H_1 with

$$(1/4)e^{N^{1/3}}Z_0 \leq |Z_1| \leq (3/4)e^{N^{1/3}}Z_0$$

and with

$$|U_1(Z_1)| = 1 + o(1), \qquad |U_2(Z_1)| = 1 + o(1), \tag{22}$$

and such that F_1F_2 has no zeros in the region

$$\{Z\colon |\log(Z/Z_1)| \leq N^{1/3}(r_m)^{-2M}\}.$$

Let v_m be the pre-image of Z_1 in H_0. Since, by (12) and (13),

$$Z^{-1}dZ/dz = (1 + o(1))(N+2)/2z$$

on $D(z_0, 1)$, it follows that $E = f_1f_2$ has no zeros ζ with

$$|\log(\zeta/v_m)| \leq \lambda, \qquad \lambda = N^{1/3}(r_m)^{-2M}(N+2)^{-1} \geq (r_m)^{-c}.$$

In the last inequality we have used (11).

It remains only to estimate $E(v_m)$. Since

$$\begin{aligned} 1 = W(f_1, f_2) &= (C_1D_2 - C_2D_1)W(u_1, u_2) \\ &= (C_1D_2 - C_2D_1)(2i + o(1)) \end{aligned}$$

in H_0, it follows from (20) and Lemma 2 that either C_1 and D_2 each have modulus at most M^{-2}, or C_2 and D_1 each have modulus at most M^{-2}. We assume without loss of generality that the latter is the case. We therefore have

$$C_1C_2 = o(1), \qquad D_1C_2 = o(1), \qquad D_1D_2 = o(1),$$
$$C_1D_2 = (1/2i)(1 + o(1)).$$

Further,

$$\begin{aligned} E &= (C_1u_1 + D_1u_2)(C_2u_1 + D_2u_2) \\ &= (C_1D_2 + D_1C_2)u_1u_2 + C_1C_2u_1^2 + D_1D_2u_2^2 \end{aligned}$$

so that using (22) we have

$$\begin{aligned} E(v_m) &= (1 + o(1))(1/2i)u_1(v_m)u_2(v_m) \\ &= (1 + o(1))(1/2i)A(v_m)^{-1/2} \end{aligned}$$

and Lemma 1 is proved.

3. PROOF OF THEOREMS 1 AND 2

Suppose that K, M, A, E, r_m are as in the statement of Theorem 1 and that v_m and M_1, M_2 are as in Lemma 1. Let $\phi(z)$ map the unit disc $\Delta = B(0,1)$ conformally onto the logarithmic rectangle $U = \{w\colon |\log|w|| < (1/4)\log K, |\arg w| < \pi\}$, with $\phi(0) = 1$, and let

$$h(z) = h_m(z) = v_m \phi(z)^2. \tag{23}$$

Defining

$$B(z) = A(h(z)), \qquad F(z) = E(h(z)), \tag{24}$$

we have, by (2),

$$(h')^2 F^{-2} = (F'/F)^2 - 2F''/F + 2(h''/h')(F'/F) - 4(h')^2 B \tag{25}$$

for z in Δ. Since A has finite order we have, by (7),

$$\log^+|B(z)| + \log^+\log^+|F(z)| \le (r_m)^d \tag{26}$$

for z in Δ, using d to denote a positive constant not depending on m, not necessarily the same at each occurrence. Using (6) and (26), we have

$$\log^+ T(r, 1/F) \le \log^+ T(r, F) + (r_m)^d \le (r_m)^d, \qquad 0 < r < 1. \tag{27}$$

Since E has no zeros in the disc $B(v_m, (r_m)^{-M_2})$, and at most $(r_m)^M$ zeros in the annulus $\Omega(r_m, K)$, we have

$$N(r, 1/F) \le (r_m)^d, \qquad 0 < r < 1. \tag{28}$$

Choose s_1 in $(0,1)$ such that the image of the disc $B(0, s_1)$ under ϕ contains the arc $|u| = 1$, $|\arg u| \le \pi/2$, and define s_j for $2 \le j \le 4$ by $s_j = (1 + s_{j-1})/2$. Again, since F has at most $2(r_m)^M$ zeros in Δ, we may choose r with $s_2 \le r \le s_3$ such that F has no zeros ζ with $||\zeta| - r| < r_m^{-2M}$. Choose R with $s_4 < R < 1$ such that F has no zeros on $|\zeta| = R$ and apply the differentiated Poisson–Jensen formula [5, p. 22] to F in $B(0, R)$ in order to estimate $F'(z)/F(z)$ and $(F'/F)'(z)$, with $|z| = r$. This gives

$$\begin{aligned} &|F'(z)/F(z)| + |(F'/F)'(z)| \\ &\qquad \le (r_m)^d + (r_m)^d (m(R, F) + m(R, 1/F)), \qquad |z| = r, \end{aligned}$$

and so, using (27),

$$m(r, F'/F) + m(r, F''/F) \le (r_m)^d.$$

Using (23), (24), (25), (26), and (28), we now have

$$T(r, 1/F) \leq (r_m)^d$$

and hence, using (6) again,

$$T(r, F) \leq (r_m)^d,$$

from which it follows that

$$\log|F(z)| \leq (r_m)^d, \qquad |z| \leq s_1,$$

and, by the choice of s_1,

$$\log|E(w)| \leq (r_m)^d, \qquad |w| = |v_m|.$$

Since the sequence $|v_m|$ satisfies $K^{-1/4} r_m < |v_m| < K^{1/4} r_m$, we obtain (4) and Theorem 1. If, in addition, we have (5), it follows at once that E has finite order, and Theorem 2 is proved.

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